MAT 2384 3X Assignment # 1: Solutions

1. $y' = 2x\sqrt{1-y^2}$, $y(0) = 1$

this equation is obviously separable $\dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx$

integrate on both sides $\displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx + C$

to get $\arcsin(y) = x^2 + C$ or $\boxed{y = \sin(x^2+C)}$ (general solution)

then $y(0)=1 \Rightarrow 1 = \sin(0+c) = \sin(c) \Rightarrow C = \pi/2$

$\therefore$ the <u>unique solution</u> is $\boxed{y = \sin(x^2 + \pi/2)}$

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2. $(2x\cos y + y^2)\,dx + (2xy - x^2\sin y)\,dy = 0$, $y(1) = \pi$
(not separable)

$M(x,y) = 2x\cos y + y^2$ $\quad M_y = -2x\sin y + 2y$
$N(x,y) = 2xy - x^2\sin y$ $\quad N_x = 2y - 2x\sin y$
$\Big\}$ $M_y = N_x$ so DE is exact

$F(x,y) = \int M(x,y)\,dx + g(y)$ $\quad \left(\text{or } \int N(x,y)\,dy + h(x)\right)$

$= \int (2x\cos y + y^2)\,dx + g(y) = x^2\cos y + xy^2 + g(y)$

then $\dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2\cos y + xy^2 + g(y)\right) = -x^2\sin y + 2xy + g'(y)$
$= N(x,y) = 2xy - x^2\sin y$

thus $g'(y) = 0 \Rightarrow g(y) = \text{constant}$, so take $g(y) = 0$

$\therefore \; F(x,y) = x^2\cos y + xy^2$

and the general solution is $\boxed{x^2\cos y + xy^2 = C}$

$y(1) = \pi \Rightarrow (1)^2\cos(\pi) + (1)(\pi)^2 = C \Rightarrow C = \pi^2 - 1$

and so the unique solution is $\boxed{x^2\cos y + xy^2 = \pi^2 - 1}$

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3. $(2x+1)\,dx + (4y+2)\,dy = 0, \quad y(1) = 1$

the DE is separable $\quad (4y+2)\,dy = -(2x+1)\,dx$

integrate on both sides $\quad \int(4y+2)\,dy = -\int(2x+1)\,dx + C$

to get $\quad 2y^2 + 2y = -x^2 - x + C$

or $\quad 2y^2 + 2y + x^2 + x = C \quad$ (general solution)

then $y(1) = 1 \Rightarrow 2(1)^2 + 2(1) + (1)^2 + (1) = C \Rightarrow C = 6$

$\therefore$ the unique solution is $\boxed{2y^2 + 2y + x^2 + x = 6}$

OR $\quad \left.\begin{array}{ll} M(x,y) = 2x+1 & M_y = 0 \\ N(x,y) = 4y+2 & N_x = 0 \end{array}\right\} \quad M_y = N_x$

DE is exact

then $F(x,y) = \int N(x,y)\,dy + h(x) \quad \left(\text{or } \int M(x,y)\,dx + g(y)\right)$

$= \int(4y+2)\,dy + h(x)$

$= 2y^2 + 2y + h(x)$

so $\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(2y^2 + 2y + h(x)\right) = h'(x) = M(x,y) = 2x+1$

so $h(x) = x^2 + x$ and $F(x,y) = 2y^2 + 2y + x^2 + x$
and the general solution is $\underline{2y^2 + 2y + x^2 + x = C}$

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4. $(x + 2y)\,dx - x\,dy = 0$, $y(1) = 5$ (not separable)

$M(x,y) = x + 2y$ $\quad M_y = 2$
$N(x,y) = -x$ $\quad N_x = -1$
$\Big\}$ $M_y \neq N_x$ so DE not exact

$M_y - N_x = 2 - (-1) = 3$, then $\dfrac{M_y - N_x}{N} = -\dfrac{3}{x}$ (function of x only)

so $\mu(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$ is integrating factor
and the DE becomes $(x^{-2} + 2x^{-3}y)\,dx - x^{-2}\,dy = 0$

then $M^*(x,y) = x^{-2} + 2x^{-3}y$, $M^*_y = 2x^{-3}$
and $N^*(x,y) = -x^{-2}$, $N^*_x = 2x^{-3}$
$\Big\}$ $M^*_y = N^*_x$ so DE exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) \quad \left(\text{or } \int N^*(x,y)\,dy + h(x)\right)$$
$$= \int (x^{-2} + 2x^{-3}y)\,dx + g(y)$$
$$= -x^{-1} - x^{-2}y + g(y)$$

then $\dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left(-x^{-1} - x^{-2}y + g(y)\right) = -x^{-2} + g'(y)$
$= N^*(x,y) = -x^{-2}$

so $g'(y) = 0$, giving $g(y) =$ constant, so take $g(y) = 0$
and $F(x,y) = -x^{-1} - x^{-2}y$

and the <u>general solution</u> is $\boxed{-x^{-1} - x^{-2}y = C}$ or $\boxed{y = Cx^2 - x}$

then $y(1) = 5 \Rightarrow 5 = C(1)^2 - (1) = C - 1 \Rightarrow C = 6$

$\therefore$ the unique solution is $\boxed{y = 6x^2 - x}$

OR since $M(x,y)$ and $N(x,y)$ are both homogeneous of degree 1, we could use the substitution

$y = ux$, $dy = u\,dx + x\,du$

(or $x = uy$, $dx = u\,dy + y\,du$)

the DE becomes $(x + 2ux)dx - x(u\,dx + x\,du) = 0$

which is $x\,dx + 2ux\,dx - ux\,dx - x^2\,du = 0$

or $x\,dx + ux\,dx - x^2\,du = 0$

or $x(1+u)\,dx - x^2\,du = 0$

which separates as $\dfrac{du}{1+u} = \dfrac{dx}{x}$

integrate on both sides $\displaystyle\int \frac{du}{1+u} = \int \frac{dx}{x} + C$

to get $\ln|1+u| = \ln|x| + C$

exponentiate both sides to get $1 + u = Kx$

or $u = Kx - 1$

which is $y/x = Kx - 1$ or $\underline{y = Kx^2 - x}$

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5. $((x+1)\ln y - y^2)dx + (x/y - 2y)dy = 0$, $y(0) = \sqrt{2}$
(not separable)

$M(x,y) = (x+1)\ln y - y^2$ $\quad M_y = (x+1)/y - 2y$

$N(x,y) = x/y - 2y$ $\quad N_x = 1/y$

$M_y \neq N_x$ not exact

but $M_y - N_x = x/y - 2y$

so $\dfrac{M_y - N_x}{N} = \dfrac{x/y - 2y}{x/y - 2y} = 1$ (function of x only)

and $\mu(x) = e^{\int dx} = e^x$ and the DE becomes

$$(e^x(x+1)\ln y - y^2e^x)dx - (xe^x/y - 2ye^x)dy = 0$$

$M^*(x,y) = e^x(x+1)\ln y - y^2e^x$ $\qquad M^*_y = e^x(x+1)/y - 2ye^x$

$N^*(x,y) = xe^x/y - 2ye^x$ $\qquad N^*_x = e^x/y + xe^x/y - 2ye^x$

$M^*_y = N^*_x$, so DE is now exact

then $F(x,y) = \int N^*(x,y)dy + h(x)$ (or $\int M^*(x,y)dx + g(y)$)

$= \int(xe^x/y - 2ye^x)dy + h(x)$

$= xe^x\ln y - y^2e^x + h(x)$

so $\dfrac{\partial F}{\partial x} = \dfrac{\partial}{\partial x}(xe^x\ln y - y^2e^x + h(x)) = e^x\ln y + xe^x\ln y - y^2e^x + h'(x)$

$= M^*(x,y) = e^x(x+1)\ln y - y^2e^x$

thus $h'(x) = 0 \Rightarrow$ we take $h(x) = 0$

and so $F(x,y) = xe^x\ln y - y^2e^x$

and the <u>general solution</u> is $\boxed{xe^x\ln y - y^2e^x = C}$

$y(0) = \sqrt{2} \Rightarrow (0)e^0\ln(\sqrt{2}) - (\sqrt{2})^2e^0 = C \Rightarrow C = -2$

$\therefore$ the <u>unique solution</u> is $\boxed{xe^x\ln y - y^2e^x = -2}$

<u>or</u> $\boxed{y^2e^x - xe^x\ln y = 2}$

3XU

A1

6

6. $(2xy^2 - y - 2y\sin x)\,dx + (6\cos x + 4x^2y - 3x)\,dy = 0, \quad y(0) = 2$

(not separable)

$M(x,y) = 2xy^2 - y - 2y\sin x \qquad M_y = 4xy - 1 - 2\sin x$

$N(x,y) = 6\cos x + 4x^2y - 3x \qquad N_x = -6\sin x + 8xy - 3$

$\Big\}$ $M_y \neq N_x$ not exact

$M_y - N_x = 4\sin x + 2 - 4xy$

so $\dfrac{M_y - N_x}{M} = \dfrac{4\sin x + 2 - 4xy}{2xy^2 - y - 2y\sin x} = \dfrac{-2(2xy - 1 - 2\sin x)}{y(2xy - 1 - 2\sin x)} = \dfrac{-2}{y}$

(a function of y only)

then $\mu(y) = e^{-\int -\frac{2}{y}dy} = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$ and the DE becomes

$$(2xy^4 - y^3 - 2y^3\sin x)\,dx + (6y^2\cos x + 4x^2y^3 - 3xy^2)\,dy = 0$$

$M^*(x,y) = 2xy^4 - y^3 - 2y^3\sin x \qquad M^*_y = 8xy^3 - 3y^2 - 6y^2\sin x$

$N^*(x,y) = 6y^2\cos x + 4x^2y^3 - 3xy^2 \qquad N^*_x = -6y^2\sin x + 8xy^3 - 3y^2$

$M^*_y = N^*_x$ and DE has been made exact

$F(x,y) = \int M^*(x,y)\,dx + g(y) \qquad \left(\text{or } \int N^*(x,y)\,dy + h(x)\right)$

$= \int (2xy^4 - y^3 - 2y^3\sin x)\,dx + g(y)$

$= x^2y^4 - xy^3 + 2y^3\cos x + g(y)$

thus $\dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left(x^2y^4 - xy^3 + 2y^3\cos x + g(y)\right)$

$= 4x^2y^3 - 3xy^2 + 6y^2\cos x + g'(y)$

$= N^*(x,y) = 6y^2\cos x + 4x^2y^3 - 3xy^2$

so $g'(y) = 0 \Rightarrow$ we take $g(y) = 0$
and $F(x,y) = x^2y^4 - xy^3 + 2y^3\cos x$

and the general solution is $\boxed{x^2y^4 - xy^3 + 2y^3\cos x = C}$

$y(0) = 2 \Rightarrow (0)^2(2)^4 - (0)(2)^3 + 2(2)^3\cos(0) = C \Rightarrow C = 16$

$\therefore$ the unique solution is $\boxed{x^2y^4 - xy^3 + 2y^3\cos x = 16}$

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7. we want x such that $f(x) = x^3 + 8x - 7 = 0$
so $8x = 7 - x^3$ or $x = \dfrac{7-x^3}{8}$

we take $g(x) = \dfrac{7-x^3}{8}$

then $\left|g'(x)\right| = \left|-\dfrac{3x^2}{8}\right| = \dfrac{3}{8}x^2 \le \dfrac{3}{8} < 1$ on $[0,1]$

$\therefore$ the sequence generated by $x_{n+1} = g(x_n)$ will converge

$x_0 = 0.75$ $\quad x_1 = g(x_0) = \dfrac{7-(0.75)^3}{8} = 0.82227$

$x_2 = g(x_1) = \dfrac{7-(0.82227)^3}{8} = 0.80551$

$x_3 = g(x_2) = \dfrac{7-(0.80551)^3}{8} = 0.80967$

$x_4 = g(x_3) = \dfrac{7-(0.80967)^3}{8} = 0.80865$

3X4
A1
8

$$x_5 = g(x_4) = \frac{7 - (0.80865)^2}{8} = 0.80890$$

$$x_6 = g(x_5) = \frac{7 - (0.80890)^3}{8} = 0.80884$$

$$x_7 = g(x_6) = \frac{7 - (0.80884)^3}{8} = 0.80885$$

$$x_8 = g(x_7) = \frac{7 - (0.80885)^3}{8} = 0.80885 = x_7$$

$\therefore$ stop

$\therefore$ the root is $\underline{0.80885}$ (to 5 decimal places)

(Check: $f(0.80885) = (0.80885)^3 + 8(0.80885) - 7 \simeq -1.9 \times 10^{-5}$)